$\mathbb{Z}_2^n$ grading of the classical Lie algebras.

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The $\mathbb{Z}_2^n$ gradings of the classical Lie algebras are described. To elucidate the grading, the classical Lie algebras are expressed in terms of matrix algebras, $\mathrm{mat}(N;\mathbb{F})$, where $\mathbb{F}$ is one of eight fields or Clifford algebras which carry gradings ranging from zero to $\mathbb{Z}_2^3$. When the matrix dimension N is divisible by a power of two, the Lie algebras have higher gradings which may be expressed in terms of higher Clifford algebras: these are also described.

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1 Introduction

In Weyl's original classification of the classical Lie algebras [1], scant regard was paid to their graded structure. The introduction of supersymmetry and Lie superalgebras to physics [2, 3, 4] led to renewed scrutiny of the properties of Lie algebras concomitant with their grading, and in 1978 Rittenberg, Wyler and Lukierski discovered the generalized Lie algebras and superalgebras, also known as Lie color algebras and Lie color superalgebras [5, 6, 7]. Rittenberg *et al.* demonstrated how to construct a Lie color algebra from a Lie algebra with a grading, and Scheunert subsequently proved that all Lie color algebras can be obtained in this manner [8].

A knowledge of the grading of a Lie algebra is required for the construction of the associated Lie color algebras: this provided the impetus for the present study of the $\mathbb{Z}_2^n$ gradings of the classical Lie algebras.

The paper is organized as follows. Sec. 2 contains introductory remarks on graded algebras and Clifford algebras and also serves to define the notation.

Sec. 3 describes a graded classification of the classical Lie algebras, based on their description in terms of matrix algebras over one of eight graded fields or Clifford algebras.

Sec. 4 describes higher gradings of Lie algebras, for example the grading of $O(N;\mathbb{R})$ when N is divisible by a power of two, and recasts these Lie algebras in terms of matrix algebras over higher Clifford algebras. Sec. 5 presents concluding remarks.

2 Conventions on graded algebras

Throughout this work Γ denotes a finite *abelian* group. The groups of principal relevance are $\mathbb{Z}_2$ and $\mathbb{Z}_2^n$, the direct sum of n copies of $\mathbb{Z}_2$. A convenient representation of $\mathbb{Z}_2$ is the set $\{0,1\}$ with addition modulo 2, and the corresponding representation of $\mathbb{Z}_2^n$ consists of the 2^n n-tuples comprised of 1's and 0's.

A *Γ-graded vector space* consists of a vector space V equipped with a partition into subspaces, each of which is labelled by an element of Γ,

$$V = \bigoplus_{\gamma\in\Gamma} V_\gamma$$

A *Γ-graded algebra* consists of an algebra A which is a graded vector space,

$$A = \bigoplus_{\gamma\in\Gamma} A_\gamma$$

and additionally satisfies,

$$A_\alpha A_\beta \subset A_{\alpha+\beta}, \qquad \forall\,\alpha,\beta\in\Gamma$$

It follows that if A has a unit element then it belongs to A_0.

Consider a graded algebra equipped with an adjoint, denoted by †. In addition to the usual conditions for an anti-involution,

$$(xy)^\dagger = y^\dagger x^\dagger \qquad \forall x,y\in A$$
$$(x^\dagger)^\dagger = x$$

the adjoints employed in this work are also required to preserve the grading,

$$(A_\alpha)^\dagger \subset A_\alpha \tag{2.1}$$

The *Clifford algebra* $Cl(d_+,d_-;\mathbb{R})$ is a real algebra of dimension $2^{\,d_++d_-}$, generated by d_++d_- basis elements e_i satisfying,

$$e_i^2 = \begin{cases} 1, & 1\le i\le d_+ \\ -1, & d_+ < i \le d_++d_- \end{cases}$$
$$e_ie_j + e_je_i = 0, \qquad i\neq j$$
$$e_1e_2\ldots e_{d_++d_-} \neq \pm 1$$

e_{ijk} is a shorthand notation for $e_ie_je_k$. Each Clifford algebra has several adjoint operators, collectively denoted by †. The first adjoint, *Clifford Reversion*, reverses the order of the Clifford product,

$$\widetilde{e_ie_je_k} := e_ke_je_i = -e_{ijk} \tag{2.2a}$$

while *Clifford Conjugation* also introduces a sign-change,

$$\overline{e_ie_je_k} := (-e_k)(-e_j)(-e_i) = e_{ijk} \tag{2.2b}$$

Additional adjoints can be obtained by changing the sign of selected e_i's:

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Notation. $x^{\dagger(\boldsymbol{a}_+,\boldsymbol{a}_-)}$ denotes the adjoint obtained by composing Clifford Reversion with the following sign-change,

$$(e_ie_je_k)^{\dagger(a_+,a_-)} := e_k^{\dagger(a_+,a_-)}e_j^{\dagger(a_+,a_-)}e_i^{\dagger(a_+,a_-)}$$
$$e_i^{\dagger(a_+,a_-)} := \begin{cases} e_i, & 1\le i\le a_+ \\ -e_i, & a_+<i\le d_+ \\ e_i, & d_+<i\le d_++a_- \\ -e_i, & d_++a_-<i\le d_++d_- \end{cases} \tag{2.2c}$$

Thus $x^{\dagger(0,0)}=\bar{x}$ and $x^{\dagger(d_+,d_-)}=\tilde{x}$.

$Cl(d_+,d_-;\mathbb{R})$ has a natural $\mathbb{Z}_2^{d_++d_-}$ grading. The grade associated with e_{ijk} is the (d_++d_-)-tuple with 1's in positions i,j,k and 0's in the remaining positions. $e_{ijk}^\dagger$ has the same grade as e_{ijk}, as required by Eq. (2.1).

Example. $e_{124}\in Cl(5,0;\mathbb{R})$ has grade $(1,1,0,1,0)$.

If algebras A and B have gradings $\mathbb{Z}_2^{n_a}$ and $\mathbb{Z}_2^{n_b}$, then the tensor product $A\otimes B$ has a $\mathbb{Z}_2^{n_a+n_b}$ grading,

$$A_\alpha\otimes B_\beta =: (A\otimes B)_{\alpha\oplus\beta} \tag{2.3}$$

For the purposes of grading, all algebras are regarded as real algebras. Hence Clifford algebras over the complex field, $Cl(d;\mathbb{C})$, are regarded as tensor products of real algebras,

$$Cl(d_++d_-;\mathbb{C}) \cong \mathbb{C}\otimes_{\mathbb{R}} Cl(d_+,d_-;\mathbb{R}) \tag{2.4a}$$

where $\mathbb{C}\cong Cl(0,1;\mathbb{R})$ has a $\mathbb{Z}_2^1$ grading. Adjoints act on $Cl(1;\mathbb{C})$ as,

$$(\mathbb{C}\otimes_{\mathbb{R}} Cl(d_+,d_-;\mathbb{R}))^\dagger = \mathbb{C}\otimes_{\mathbb{R}} Cl(d_+,d_-;\mathbb{R})^\dagger \tag{2.4b}$$

Adjoints which include complex conjugation can also be defined, but are not employed in this work.

Consider the *matrix algebra* $\mathbb{F}(D)$ of $D\times D$ matrices over $\mathbb{F}$, where $\mathbb{F}$ is a Clifford algebra. $\mathbb{F}(D)$ may be regarded as a real tensor product,

$$\mathbb{F}(D)\cong\mathbb{R}(D)\otimes_{\mathbb{R}}\mathbb{F} \tag{2.5a}$$

and it follows that $\mathbb{F}(D)$ inherits the grading of $\mathbb{F}$.

The adjoint of the tensor-product $\mathbb{F}(D)$ is the tensor product of the adjoints for $\mathbb{R}(D)$ and $\mathbb{F}$,

$$(\mathbb{R}(D)\otimes_{\mathbb{R}}\mathbb{F})^\dagger = \mathbb{R}(D)^\dagger\otimes_{\mathbb{R}}\mathbb{F}^\dagger \tag{2.5b}$$

and conforms with Eq. (2.1). The adjoints for $\mathbb{F}$ are given by Eq. (2.2). The adjoint of $\mathbb{R}(D)$ is [9],

$$\mathbb{R}(D)^\dagger = g^{-1}\mathbb{R}(D)^t g \tag{2.5c}$$

where $\mathbb{R}(D)^t$ denotes the matrix transpose and $g\in\mathbb{R}(D)$ is the metric. $\mathbb{R}(D)$ admits adjoints with both skew-symmetric and symmetric non-degenerate metrics. The symmetric metrics may be either Euclidean, $g=I$, or pseudo-Euclidean with signature (D_+,D_-), $g=I_{D_+,D_-}$.

Notation.

$\mathbb{F}(D_+,D_-)$	$\mathbb{F}(D_++D_-)$ equipped with $g=I_{D_+,D_-}$
$(\mathbb{F}(D_+,D_-),\dagger)$	$\mathbb{F}(D_+,D_-)$ equipped with an adjoint, cf Eq. (2.5b)
$\widetilde{\mathbb{F}}(D_+,D_-)$	$(\mathbb{F}(D_+,D_-),\dagger)$ where $\mathbb{F}^\dagger=\tilde{\mathbb{F}}$.
$\overline{\mathbb{F}}(D_+,D_-)$	$(\mathbb{F}(D_+,D_-),\dagger)$ where $\mathbb{F}^\dagger=\bar{\mathbb{F}}$.

3 $\mathbb{F}$-grading of the classical Lie algebras

3.A Graded classification

Each classical Lie algebra (LA) may be represented as a subalgebra of $\mathbb{F}(D)$, where $\mathbb{F}$ is a field or more generally a Clifford algebra. The LA then inherits the $\mathbb{Z}_2^n$ grading of $\mathbb{F}$. The purpose of this section is thus to find an $\mathbb{F}$ with the highest possible grading for each LA. The classical LA's associated with each $\mathbb{F}$ are described in this section, and the defining equations for each LA are presented in Sec. 3 B.

Table I lists the $\mathbb{F}$'s employed in this work, together with isomorphic Clifford algebras and their $\mathbb{Z}_2^n$ gradings.

The fields,

$$\mathbb{F}\in\{\mathbb{R},\mathbb{C}\} \tag{3.1a}$$

permit the classical LA's to be classified as [10]: orthogonal $O(D_+,D_-;\mathbb{F})$, unitary $U(D_+,D_-;\mathbb{F})$, symplectic $Sp(2D;\mathbb{F})$, general-linear $Gl(D;\mathbb{F})$, unitary-symplectic $USp(2D_+,2D_-;\mathbb{C})$, $O\divideontimes(2D;\mathbb{C})$ and $U\divideontimes(2D;\mathbb{C})$.

If Eq. (3.1a) is augmented with the quaternion field,

$$\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}\cup\{\mathbb{H}\} \tag{3.1b}$$

then USp, $O\divideontimes$ and $U\divideontimes$ may be reclassified as unitary, orthogonal and general-linear algebras over $\mathbb{H}$ [10], and this yields a higher grading than Eq. (3.1a).

Example. $U\divideontimes(2D;\mathbb{C})\cong Gl(D;\mathbb{H})$ and hence admits a $\mathbb{Z}_2^2$ grading, versus a $\mathbb{Z}_2^1$ grading using Eq. (3.1a).

If Eq. (3.1b) is augmented with three Clifford algebras,

$$\mathbb{F}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}\cup\{{}^2\mathbb{R},{}^2\mathbb{C},{}^2\mathbb{H}\} \tag{3.1c}$$

then the general-linear algebras may be reclassified as unitary [9], and this yields a higher grading than Eq. (3.1b).

Example. $Gl(D;H)\cong U(D;{}^2\mathbb{H})$ and hence admits a $\mathbb{Z}_2^3$ grading, versus a $\mathbb{Z}_2^2$ grading using Eq. (3.1b).

The six $\mathbb{F}$'s in *Eq.* (3.1c) arise naturally in the context of Clifford algebras, because every Clifford algebra is isomorphic to $\mathbb{F}(2^n)$ for one such $\mathbb{F}$ [11]. However, in order to expose the $\mathbb{Z}_2^n$ grading of all the classical LA's, we have found it necessary to supplement *Eq.* (3.1c) with two additional Clifford algebras, the split-quaternions $\mathbb{H}''$ and the biquaternions $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{H}''$,

$$\mathbb{F}\in\{\mathbb{R},\mathbb{C},\mathbb{H},{}^2\mathbb{R},{}^2\mathbb{C},{}^2\mathbb{H}\}\cup\{\mathbb{H}'',\mathbb{C}\otimes_{\mathbb{R}}\mathbb{H}''\} \tag{3.1d}$$

TABLE I: Classification of the Classical Lie Algebras.[a]

$\mathbb{F}$	$\mathbb{Z}_2^n$ grading	$\mathbb{F}^\dagger$ (Eq.'s (2.5) and (3.2)) Clifford Reversion	Clifford Conjugation
$\mathbb{R} \cong Cl(0,0;\mathbb{R})$	—	$O(D_+,D_-;\mathbb{R})$	—
$\mathbb{C} \cong Cl(0,1;\mathbb{R})$	$\mathbb{Z}_2^1$	$O(D;\mathbb{C})$	$U(D_+,D_-;\mathbb{C})$
${}^2\mathbb{R} \cong Cl(1,0;\mathbb{R})$	$\mathbb{Z}_2^1$	—	$U(D;{}^2\mathbb{R}) \cong Gl(D;\mathbb{R})$
$\mathbb{H} \cong Cl(0,2;\mathbb{R})$	$\mathbb{Z}_2^2$	$O(D;\mathbb{H}) \cong O*(2D;\mathbb{C})$	$U(D_+,D_-;\mathbb{H}) \cong USp(2D_+,2D_-;\mathbb{C})$
$\mathbb{H}'' \cong Cl(2,0;\mathbb{R})^b$	$\mathbb{Z}_2^2$	—	$U(D;\mathbb{H}'') \cong Sp(2D;\mathbb{R})$
${}^2\mathbb{C} \cong Cl(1;\mathbb{C})^c$	$\mathbb{Z}_2^2$	—	$U(D;{}^2\mathbb{C}) \cong Gl(D;\mathbb{C})$
${}^2\mathbb{H} \cong Cl(0,3;\mathbb{R})$	$\mathbb{Z}_2^3$	$U(D;{}^2\mathbb{H}) \cong U*(2D;\mathbb{C})^d$	—
$\mathbb{C}\otimes_{\mathbb{R}}\mathbb{H}'' \cong Cl(3,0;\mathbb{R})^e$	$\mathbb{Z}_2^3$	—	$O_{\mathbb{C}}U_{\mathbb{H}''}(D;\mathbb{C}\otimes_{\mathbb{R}}\mathbb{H}'') \cong Sp(2D;\mathbb{C})$ [f]

[a] A blank in the table indicates that the corresponding Lie algebra is superfluous to the classification.
[b] $\mathbb{H}'' \cong Cl(2,0;\mathbb{R}) \cong \mathbb{R}(2)$
[c] ${}^2\mathbb{C} \cong Cl(1;\mathbb{C}) \cong Cl(0,1;\mathbb{R}) \otimes_{\mathbb{R}} Cl(1,0;\mathbb{R})$. $U(D;{}^2\mathbb{C})$ may be obtained using any of the following adjoints: (i) Clifford Conjugation on $Cl(1;\mathbb{C})$. This is equivalent to Clifford Conjugation on $Cl(1,0;\mathbb{R})$ and the Identity operator on $Cl(0,1;\mathbb{R})$. (ii) Clifford Conjugation on both $Cl(1,0;\mathbb{R})$ and $Cl(0,1;\mathbb{R})$.
[d] $U(D;{}^2\mathbb{H}) \cong Gl(D;\mathbb{H}) \cong U*(2D;\mathbb{C})$
[e] $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{H}'' \cong Cl(3,0;\mathbb{R}) \cong \mathbb{C}(2)$
[f] The notation $O_{\mathbb{C}}U_{\mathbb{H}''}$ indicates that Clifford Conjugation on $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{H}''$ is equivalent to the Identity operator on $\mathbb{C}$ and Clifford Conjugation on $\mathbb{H}''$. The Identity operator coincides with Clifford Reversion on $\mathbb{C} \cong Cl(0,1;\mathbb{R})$.

Eq. (3.1d) permits the symplectic algebras to be reclassified, thereby yielding a higher grading than Eq. (3.1c).

Example. $Sp(2D;\mathbb{R}) \cong U(D;\mathbb{H}'')$ and hence admits a $\mathbb{Z}_2^2$ grading, versus no grading using Eq. (3.1c).

Table I contains the classification of the classical LA's with respect to Eq. (3.1d). A blank in the table indicates that the corresponding LA is superfluous to the classification. For instance, neither $O(D;\mathbb{H}'') \cong O(2D;\mathbb{R}) \subset O(D_+,D_-;\mathbb{R})$ nor $O(D;{}^2\mathbb{R}) \cong O(D,0;\mathbb{R})\oplus O(D,0;\mathbb{R})$ is needed for the classification.

3.B Defining equations

The defining equations for the classical LA's take the form,

$$L^\dagger + L = 0 \tag{3.2a}$$

where,

$$L \subset \widetilde{\mathbb{F}}(D_+,D_-)\ , \quad \text{or} \quad L \subset \overline{\mathbb{F}}(D_+,D_-) \tag{3.2b}$$

The adjoint (†) is given by Eq. (2.5), and involves the adjoint of $\mathbb{F}$ and the metric g. The adjoint of $\mathbb{F}$ associated with each LA — Reversion or Clifford Conjugation — is indicated in Table I.

In the conventional classification of the LA's, skew-symmetric metrics are employed for the symplectic groups. In contrast, under the current classification g is always symmetric. $g = I_{D_+,D_-}$, the pseudo-Euclidean metric, for $O(D_+,D_-;\mathbb{R})$, $U(D_+,D_-;\mathbb{C})$ and $U(D_+,D_-;\mathbb{H})$; the other LA's are independent of the signature of the metric, so we are free to choose $g = I$.

Remark. Eq. (3.2) defines the automorphism algebras for $\widetilde{\mathbb{F}}(D_+,D_-)$ and $\overline{\mathbb{F}}(D_+,D_-)$,

$$L = \mathfrak{aut}\big(\mathbb{F}(D_+,D_-),\dagger\big) \tag{3.2c}$$

L inherits the grading of $\mathbb{F}$: this follows from Eq. (2.1).

Example. Referring to table I,

$$\mathfrak{aut}\big(\widetilde{\mathbb{H}}(D_+,D_-)\big) = O(D_+ + D_-;\mathbb{H})$$
$$\mathfrak{aut}\big(\overline{\mathbb{H}}(D_+,D_-)\big) = U(D_+,D_-;\mathbb{H})$$

Example 3.1 *(*Definition of $U(D;\mathbb{H}'')$*).* Consider the LA obtained from Eq. (3.2) with $\mathbb{F} = \mathbb{H}'' \cong Cl(2,0;\mathbb{R})$, $g = I$ and † = Clifford Conjugate. L may be expressed as a tensor product (Eq. (2.5a)),

$$L = L_1 \otimes 1 + L_{e_1} \otimes e_1 + L_{e_2} \otimes e_2 + L_{e_{12}} \otimes e_{12} \tag{3.3}$$

where $L_i \subset \mathbb{R}(D)$. Eq.'s (2.2b), (2.5c) and (3.2a) then yield,

$$\begin{aligned} 0 = L + L^\dagger = (L_1 + L_1^t)\otimes 1 + (L_{e_1} - L_{e_1}^t)\otimes e_1 \\ + (L_{e_2} - L_{e_2}^t)\otimes e_2 + (L_{e_{12}} - L_{e_{12}}^t)\otimes e_{12} \end{aligned}$$

Consequently L_1 is skew-symmetric and L_{e_1}, L_{e_2} and $L_{e_{12}}$ are symmetric. More generally, for any classical LA,

$$L_i^\dagger = \pm L_i$$

Let symmetric matrices be denoted by S_i and skew-symmetric matrices by A_i.

A matrix-representation of L may be constructed, using Eq. (3.3) in conjunction with a matrix-representation of $\mathbb{F}$. An $\mathbb{R}(2)$ representation of $\mathbb{F}$ is given by,

$$1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right),\ e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right),\ e_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right),\ e_{12} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \tag{3.4}$$

leading to an $\mathbb{R}(2D)$ representation of L,

$$L = \left(\begin{array}{c|c} A_1 + S_{e_1} & S_{e_2} + S_{e_{12}} \\ \hline S_{e_2} - S_{e_{12}} & A_1 - S_{e_1} \end{array}\right)$$

This matches the representation of $Sp(2D;\mathbb{R})$ [10].

Remark 3.1. The grading of $U(1;\mathbb{H}'')$ and more generally of LA's with $D = 1$ or $D_+ + D_- = 1$ requires special consideration. The 1×1 skew-symmetric matrices (A_i above) are zero, and this may reduce the grading. For instance, in $O(1;\mathbb{H})$ only $L_{e_{12}}$ is non-zero. The LA resides inside a subalgebra of $\mathbb{H}$, namely $(1, e_{12})$, and the subalgebra carries a $\mathbb{Z}_2^1$ grading, versus $\mathbb{Z}_2^2$ for $\mathbb{H}$.

Nomenclature

The traditional nomenclature for the symplectic algebras, $Sp(2D;\mathbb{R})$ and $Sp(2D;\mathbb{C})$, is not appropriate for the description of these algebras in terms of $\mathbb{F} = \mathbb{H}''$ and $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}''$: a new notation is desirable.

Since the classical LA's are automorphism algebras, they may be specified as $\mathfrak{aut}\big(\widetilde{\mathbb{F}}(D_+, D_-)\big)$ or $\mathfrak{aut}\big(\overline{\mathbb{F}}(D_+, D_-)\big)$.

Alternatively, the designation of each LA as $O(D_+, D_-;\mathbb{F})$ or $U(D_+, D_-;\mathbb{F})$ is in common use for the six $\mathbb{F}$'s in Eq. (3.1c) [9], so to complete the classification we need only consider the remaining two $\mathbb{F}$'s: $\mathbb{H}''$ and $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}''$.

$U(D;\mathbb{H}'')$ is defined analogously to $U(D, 0;\mathbb{H})$; both LA's may also be specified as $\mathfrak{aut}\big(\overline{\mathbb{F}}(D, 0)\big)$, $\mathbb{F} = \mathbb{H}$ or $\mathbb{H}''$.

In order to obtain an alternative notation for $\mathfrak{aut}\big(\overline{\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}''}(D, 0)\big)$, let us first express the adjoint as a tensor product,

$$\overline{\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}''} = \widetilde{\mathbb{C}} \otimes_{\mathbb{R}} \overline{\mathbb{H}''}$$

Thus Clifford Conjugation of the tensor product is equivalent to Reversion of $\mathbb{C}$ tensored with Clifford Conjugation of $\mathbb{H}''$. Since $\mathfrak{aut}\big(\widetilde{\mathbb{C}}(D, 0)\big) \cong O(D;\mathbb{C})$ and $\mathfrak{aut}\big(\overline{\mathbb{H}''}(D, 0)\big) \cong U(D;\mathbb{H}'')$, a logical but cumbersome designation for $\mathfrak{aut}\big(\overline{\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}''}(D, 0)\big)$ is $O_{\mathbb{C}}U_{\mathbb{H}''}(D;\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}'')$.

Remark. Reversion has no effect on $\mathbb{C} \cong Cl(0, 1;\mathbb{R})$, since $\tilde{1} = 1$ and $\tilde{e_1} = e_1$. Consequently $O(D;\mathbb{C}) \cong \mathbb{C} \otimes_{\mathbb{R}} O(D, 0;\mathbb{R})$, and $O_{\mathbb{C}}U_{\mathbb{H}''}(D;\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}'') \cong \mathbb{C} \otimes_{\mathbb{R}} U(D;\mathbb{H}'')$.

4 Higher grading

4.A Tensor product of graded algebras

In Sec. 3 each classical LA was expressed as a subalgebra of $\mathbb{F}(D)$ and consequently inherited the grading of $\mathbb{F}$. $\mathbb{F}(D)$ may be regarded as a tensor product of $\mathbb{R}(D)$ and $\mathbb{F}$, (Eq. (2.5a)), and both components may carry a grading. The total grading is then the direct sum of the gradings for each term (cf Eq. (2.3)),

$$\Gamma(\mathbb{F}(D)) = \Gamma(\mathbb{R}(D)) \oplus \Gamma(\mathbb{F})$$

Proposition 4.1.

$$\Gamma(\mathbb{F}(D_+\, 2^n,\, D_-\, 2^n)) = \begin{cases} \mathbb{Z}_2^{2n} \oplus \Gamma(\mathbb{F}), & D_+ \neq D_- \\ \mathbb{Z}_2^{2(n+1)} \oplus \Gamma(\mathbb{F}), & D_+ = D_- \end{cases}$$

and this grading is inherited by the classical LA's.

Proof. $\mathbb{R}(D_+\, 2^n,\, D_-\, 2^n)$ and its metric may be expressed as tensor products,

$$\mathbb{R}(D_+\, 2^n,\, D_-\, 2^n) \cong \mathbb{R}(D_+, D_-) \underset{n}{\otimes} \mathbb{R}(2) \tag{4.1}$$

$$I_{D_+\, 2^n,\, D_-\, 2^n} \cong I_{D_+, D_-} \underset{n}{\otimes} I_2 \tag{4.2}$$

where $\underset{n}{\otimes}\mathbb{R}(2)$ denotes the tensor product of n copies of $\mathbb{R}(2)$.

$\mathbb{R}(2)$ equipped with matrix transpose is isomorphic to $Cl(2, 0;\mathbb{R})$ equipped with Reversion (cf Eq. (3.4)),

$$\big(\mathbb{R}(2), \text{Transpose}\big) \cong \big(Cl(2, 0;\mathbb{R}), \text{Reversion}\big) \tag{4.3}$$

Substituting Eq.'s (4.1)—(4.3) into Eq. (2.5) yields,

$$\mathbb{F}(D_+\, 2^n,\, D_-\, 2^n) \cong \mathbb{R}(D_+, D_-) \underset{n}{\otimes} Cl(2, 0;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{F} \tag{4.4a}$$

$$\Big(\mathbb{R}(D_+, D_-) \underset{n}{\otimes} Cl(2, 0;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{F}\Big)^{\dagger} = g^{-1}\mathbb{R}(D_+, D_-)^t g \underset{n}{\otimes} \widetilde{Cl(2, 0;\mathbb{R})} \otimes_{\mathbb{R}} \mathbb{F}^{\dagger} \tag{4.4b}$$

where,

$$g = I_{D_+, D_-} \tag{4.4c}$$

$Cl(2, 0;\mathbb{R})$ and $\mathbb{F}$ carry $\mathbb{Z}_2^2$ and $\Gamma(\mathbb{F})$ gradings which are invariant under the adjoint, and it follows from Eq. (4.4) that $\mathbb{F}(D_+\, 2^n,\, D_-\, 2^n)$ carries a $\mathbb{Z}_2^{2n} + \Gamma(\mathbb{F})$ grading which is also invariant under the adjoint. Consequently the LA (Eq. (3.2)) inherits the grading.

For the special case $D_+ = D_-$, $\mathbb{R}(D_+, D_+)$ may be further factorized as

$$\mathbb{R}(D_+, D_+) \cong \mathbb{R}(D_+, 0) \otimes \mathbb{R}(1, 1)$$
$$I_{D_+, D_+} \cong I_{D_+, 0} \otimes I_{1,1}$$

$\mathbb{R}(1, 1)$ equipped with the adjoint $I_{1,1}^{-1}\mathbb{R}(1, 1)^t I_{1,1}$ is isomorphic to $Cl(2, 0;\mathbb{R})$ equipped with $\dagger(1, 0)$,

$$\big(\mathbb{R}(1, 1), \dagger\big) \cong \big(Cl(2, 0;\mathbb{R}), \dagger(1, 0)\big)$$
$$\mathbb{R}(D_+, D_+) \cong \mathbb{R}(D_+, 0) \otimes Cl(2, 0;\mathbb{R}) \tag{4.5a}$$

$$\big(\mathbb{R}(D_+, 0) \otimes Cl(2, 0;\mathbb{R})\big)^{\dagger} = \mathbb{R}(D_+, 0)^t \otimes Cl(2, 0;\mathbb{R})^{\dagger(1,0)} \tag{4.5b}$$

and the attendant grading is $\mathbb{Z}_2^{2(n+1)} \oplus \Gamma(\mathbb{F})$.

□

TABLE II: Equivalent adjoints for tensor products and higher Clifford algebras [a]

Tensor Product	Higher Clifford Algebra
$\bigotimes_{n_+} Cl(2,0;\mathbb{R})^{\dagger(2,0)} \bigotimes_{n_-} Cl(2,0;\mathbb{R})^{\dagger(1,0)} \otimes Cl(d_+,d_-;\mathbb{R})^{\dagger(a_+,a_-)}$	$Cl\big(d_+ + n_+ + n_-, d_- + n_+ + n_- +;\mathbb{R}\big)^{\dagger(a_+ + n_+, a_- + n_-)}$
$\bigotimes_{n_+} Cl(2,0;\mathbb{R})^{\dagger(2,0)} \bigotimes_{n_-} Cl(2,0;\mathbb{R})^{\dagger(1,0)} \otimes_{\mathbb{R}} Cl(d;\mathbb{C})^{\dagger(a)}$	$Cl\big(d + 2(n_+ + n_-);\mathbb{C}\big)^{\dagger(a+n_+ + n_-)}$

[a] The adjoint $\dagger(a_+, a_-)$ is defined in Eq. (2.2)

Example. $U(D;\, {}^2\mathbb{H})$ has a $\mathbb{Z}_2^3$ grading, and $U(D\,2^n;\, {}^2\mathbb{H})$ has a $\mathbb{Z}_2^{3+2n}$ grading.

Remark. The grading of LA's associated with $\mathbb{F}(D_+\,2^n,\, D_-\,2^n)$ may be reduced for the special case $D_+ + D_- = 1$, cf Remark 3.1.

4.B Higher Clifford algebras

In Sec. 4 A the grading of $\mathbb{F}(D_+\,2^n,\, D_-\,2^n)$ was exposed by expressing the algebra as a tensor product of graded algebras (Eq. (4.4)). In this section the tensor product is expressed in terms of a higher Clifford algebra which plays a role analogous to $\mathbb{F}$ in Sec. 3.

$\mathbb{F}(D_+\,2^n,\, D_-\,2^n)$ is first expressed as,

$$\mathbb{F}(D_+\,2^n,\, D_-\,2^n) \cong \begin{cases} \mathbb{A}(D_+, D_-), & D_+ \neq D_- \\ \mathbb{A}(D_+, 0), & D_+ = D_- \end{cases} \tag{4.6a}$$

where $\mathbb{A} \supset \mathbb{F}$ is a Clifford algebra with,

$$\Gamma(\mathbb{A}) = \begin{cases} \mathbb{Z}_2^{2n} \oplus \Gamma(\mathbb{F}), & D_+ \neq D_- \\ \mathbb{Z}_2^{2(n+1)} \oplus \Gamma(\mathbb{F}), & D_+ = D_- \end{cases} \tag{4.6b}$$

We subsequently determine (m_+, m_-) such that,

$$\big(\mathbb{F}(D_+\,2^n,\, D_-\,2^n), \dagger\big) \cong \begin{cases} \big(\mathbb{A}(D_+, D_-), \dagger(m_+, m_-)\big), & D_+ \neq D_- \\ \big(\mathbb{A}(D_+, 0), \dagger(m_+, m_-)\big), & D_+ = D_- \end{cases} \tag{4.6c}$$

It follows that the LA inherits the grading of $\mathbb{A}$ and that the LA may be defined directly in terms of a Clifford algebra with a manifest grading,

$$L = \begin{cases} \mathfrak{aut}\big(\mathbb{A}(D_+, D_-), \dagger(m_+, m_-)\big), & D_+ \neq D_- \\ \mathfrak{aut}\big(\mathbb{A}(D_+, 0), \dagger(m_+, m_-)\big), & D_+ = D_- \end{cases} \tag{4.7}$$

Lemma 4.2.

$$\bigotimes_n Cl(2,0;\mathbb{R}) \otimes Cl(d_+, d_-;\mathbb{R}) \cong Cl(d_+ + n, d_- + n;\mathbb{R}) \tag{4.8a}$$

$$\bigotimes_n Cl(2,0;\mathbb{R}) \otimes_{\mathbb{R}} Cl(d;\mathbb{C}) \cong Cl(d + 2n;\mathbb{C}) \tag{4.8b}$$

Proof. A standard expression for tensor products involving $Cl(2,0;\mathbb{R})$ is [9],

$$Cl(2,0;\mathbb{R}) \otimes Cl(d_+, d_-;\mathbb{R}) \cong Cl(d_+ + 1, d_- + 1;\mathbb{R}) \tag{4.9}$$

Eq. (4.8a) follows by induction. Eq. (4.8b) follows upon tensoring Eq (4.8a) with $\mathbb{C}$ and applying Eq. (2.4). □

Lemma 4.3. The adjoint for each tensor product in Table II is equivalent to the adjoint of the higher Clifford algebra in the same row.

Proof. Let us first consider the real Clifford algebras. Let $e_i,\ i \leq d_+ + d_-$, denote basis elements for $Cl(d_+, d_-;\mathbb{R})$, and $E_i,\ i \leq 2$, denote basis elements for $Cl(2,0;\mathbb{R})$.

The $(d_+ + d_- + 2)$ basis elements for $Cl(d_+ + 1, d_- + 1;\mathbb{R})$ are [12],

$$\{E_1 \otimes e_i,\ E_2 \otimes 1,\ E_{12} \otimes 1\}$$

Applying the tensor-product adjoint yields,

$$\Big\{E_1^{\dagger(2,0)} \otimes e_i^{\dagger(a_+,a_-)},\ E_2^{\dagger(2,0)} \otimes 1,\ E_{12}^{\dagger(2,0)} \otimes 1\Big\} = \Big\{E_1 \otimes e_i^{\dagger(a_+,a_-)},\ E_2 \otimes 1,\ -E_{12} \otimes 1\Big\}$$

$E_2 \otimes 1$ squares to $+1$ and does not change sign under the adjoint, hence the corresponding adjoint of $Cl(d_+ + 1, d_- + 1;\mathbb{R})$ is $\dagger(a_+ + 1, a_-)$. Replacing $E_i^{\dagger(2,0)}$ with $E_i^{\dagger(1,0)}$ yields $\dagger(a_+, a_- + 1)$, and the first row of Table II follows by induction on n_+ and n_-. The second row follows upon tensoring the first row with $\mathbb{C}$ and applying Eq. (2.4). □

Proposition 4.4.
For $\mathbb{F} \cong Cl(d_{\mathbb{F}+}, d_{\mathbb{F}-};\mathbb{R})$,

$$\big(\mathbb{F}(D_+\,2^n,\, D_-\,2^n), \dagger(a_{\mathbb{F}+}, a_{\mathbb{F}-})\big) \cong \begin{cases} \big(\mathbb{A}(D_+,\, D_-), \dagger(a_{\mathbb{F}+} + n,\, a_{\mathbb{F}-})\big), & D_+ \neq D_- \\ \big(\mathbb{A}(D_+,\, 0), \dagger(a_{\mathbb{F}+} + n,\, a_{\mathbb{F}-} + 1)\big), & D_+ = D_- \end{cases}$$

where,

$$\mathbb{A} = \begin{cases} Cl(d_{\mathbb{F}+} + n,\, d_{\mathbb{F}-} + n;\mathbb{R}), & D_+ \neq D_- \\ Cl(d_{\mathbb{F}+} + n + 1,\, d_{\mathbb{F}-} + n + 1;\mathbb{R}), & D_+ = D_- \end{cases}$$

For $\mathbb{F} = {}^2\mathbb{C} \cong Cl(1;\mathbb{C})$,

$$\big({}^2\mathbb{C}(D\,2^n), \dagger(a_{\mathbb{F}})\big) \cong \big(\mathbb{A}(D), \dagger(a_{\mathbb{F}} + n)\big)$$

where

$$\mathbb{A} = Cl(1+2n, \mathbb{C})$$

Proof. Eq. (4.6a) may be proved by applying Lemma 4.2 to Eq. (4.4a), where $\mathbb{F}$ and $\mathbb{A}$ are as defined in Prop. 4.4.

The remaining task is to establish the connection between the adjoints of $\mathbb{F}(D_+\, 2^n,\, D_-\, 2^n)$ and $\mathbb{A}(D_+, D_-)$ which arise in the proposition.

Setting,

$$(d_+, d_-) = (\, d_{\mathbb{F}+}\,,\, d_{\mathbb{F}-}), \quad (a_+, a_-) = (\, a_{\mathbb{F}+}\,,\, a_{\mathbb{F}-}),$$
$$(n_+, n_-) = (n, 0)$$

in the first row of Table II and then comparing with Eq. (4.4) with (D_+, D_-) set to $(1, 0)$ yields,

$$\big(\mathbb{F}(2^n), \dagger(\, a_{\mathbb{F}+}\,,\, a_{\mathbb{F}-})\big) \cong \big(\mathbb{A}, \dagger(\, a_{\mathbb{F}+} + n,\, a_{\mathbb{F}-})\big)$$

This establishes the connection between the adjoints for the real case, and Prop. 4.4 follows upon taking the tensor product of this equation with $\mathbb{R}(D_+, D_-)$.

The complex case follows upon taking the tensor product with $\mathbb{C}$ and applying Eq. (2.4). The special case, $D_+ = D_-$, also involves Eq. (4.5) and requires setting $(n_+, n_-) = (n, 1)$ in Table II. □

Example.

$$\Big({}^2\mathbb{H}(D\, 2^n), \text{Reversion}\Big) \cong \big(\mathbb{A}(D), \dagger(n, 3)\big)$$

where

$$\mathbb{A} = Cl(n, n+3; \mathbb{R})$$

Hence,

$$U(D\, 2^n;\, {}^2\mathbb{H}) = \mathfrak{aut}\big(\mathbb{A}(D), \dagger(n, 3)\big)$$

and both $\mathbb{A}$ and $U(D\, 2^n;\, {}^2\mathbb{H})$ have a $\mathbb{Z}_2^{3+2n}$ grading.

Remark 4.1. Prop. 4.4 employs non-standard adjoints of $\mathbb{A}$, ie adjoints other than Clifford Conjugation and Reversion. In some cases it is possible to replace a non-standard adjoint with a standard adjoint by exploiting the periodicity properties of the Clifford algebras:

Example 4.1. Setting $\mathbb{F} = \mathbb{R}$, $n = D_+ = 1$ and $D_- = 0$ in Prop. 4.4 yields,

$$\big(\mathbb{R}(2), \text{Transpose}\big) \cong \big(Cl(1, 1; \mathbb{R}), \dagger(1, 0)\big)$$

An alternative involving a standard adjoint is provided by $Cl(2, 0; \mathbb{R})$ equipped with Reversion,

$$\begin{aligned}\big(\mathbb{R}(2), \text{Transpose}\big) &\cong \big(Cl(1, 1; \mathbb{R}), \dagger(1, 0)\big) \\ &\cong \big(Cl(2, 0; \mathbb{R}), \text{Reversion}\big)\end{aligned}$$

Example 4.2. Setting $\mathbb{F} = \mathbb{R}$, $n = 2$, $D_+ = 1$ and $D_- = 0$ in Prop. 4.4 yields,

$$\big(\mathbb{R}(4), \text{Transpose}\big) \cong \big(Cl(2, 2; \mathbb{R}), \dagger(2, 0)\big)$$

In this case no standard adjoint can be employed, because $Cl(2, 2; \mathbb{R})$ is not isomorphic to $Cl(4, 0; \mathbb{R})$ or $Cl(0, 4; \mathbb{R})$,

$$Cl(2, 2; \mathbb{R}) \cong \mathbb{R}(4) \ncong \mathbb{H}(2) \cong Cl(4, 0; \mathbb{R}) \cong (0, 4; \mathbb{R})$$

Example 4.3. For $\mathbb{F} = {}^2\mathbb{C} \cong Cl(1; \mathbb{C})$ equipped with Clifford Conjugation, it is always possible to use a standard adjoint,

$$\overline{{}^2\mathbb{C}(D\, 2^n)} \cong \begin{cases} \overline{\mathbb{A}(D)}, & n \text{ even} \\ \widetilde{\mathbb{A}(D)}, & n \text{ odd} \end{cases} \tag{4.10}$$

where

$$\mathbb{A} = Cl(1+2n, \mathbb{C}) \tag{4.11}$$

This result is implicit in Porteous's survey of the periodicity properties of the real and complex Clifford algebras [9].

4.C Non-tensorial grading

The grading described in Sec's. 4 A and 4 B reflects the tensorial structure of the Lie algebras. It is also possible to impose a $\mathbb{Z}_2^n$ grading that is not associated with a tensorial structure. For example, a $\mathbb{Z}_2^n$ grading can be imposed on a vector space or a matrix. In this section we briefly describe such a grading.

In Sec. 4 B each classical LA was expressed as a subalgebra of $\mathbb{A}(D)$ or $\mathbb{A}(D_+, D_-)$ and consequently inherited the grading of $\mathbb{A}$. $\mathbb{A}(D)$ may be regarded as a tensor product of $\mathbb{R}(D)$ with $\mathbb{A}$, and $\mathbb{R}(D)$ may carry a non-tensorial grading. The total grading is then the direct sum of the gradings for each term (cf Eq. (2.3)),

$$\Gamma(\mathbb{A}(D)) = \Gamma(\mathbb{R}(D)) \oplus \Gamma(\mathbb{A})$$

The (D_+, D_-) metric does not affect the non-tensorial grading, and hence $\mathbb{R}(D_+, D_-)$ is equivalent to $\mathbb{R}(D_+ + D_-)$ for grading purposes.

A $\mathbb{Z}_2^1$ grading may be imposed by partitioning $\mathbb{R}(D)$ into diagonal and off-diagonal blocks. The diagonal blocks must be square but the dimensions are otherwise arbitrary,

$$\mathbb{R}(D) = \mathbb{R}(D)_0 + \mathbb{R}(D)_1\ ,$$

where,

$$\mathbb{R}(D) = \left[\begin{array}{c|c} \mathbb{R}(D)_0 & \mathbb{R}(D)_1 \\ \hline \mathbb{R}(D)_1 & \mathbb{R}(D)_0 \end{array}\right]$$

The grading is well-behaved under multiplication and invariant under transpose. It is also invariant under the adjoint (Eq. (2.5c)), because the metric is diagonal (pseudo-Euclidean) and hence does not mix matrix elements with different grades (Sec. 3 B). The grading is not compatible with a skew-symmetric metric, but in the present work only pseudo-Euclidean metrics are employed. The arbitrary subdivision may be repeated to yield a $\mathbb{Z}_2^2$ grading,

$$\mathbb{R}(D) = \mathbb{R}(D)_{00} + \mathbb{R}(D)_{01} + \mathbb{R}(D)_{10} + \mathbb{R}(D)_{11}$$

where,

$\mathbb{R}(D) =$

$\mathbb{R}(D)_{00}$	$\mathbb{R}(D)_{01}$	$\mathbb{R}(D)_{10}$	$\mathbb{R}(D)_{11}$
$\mathbb{R}(D)_{01}$	$\mathbb{R}(D)_{00}$	$\mathbb{R}(D)_{11}$	$\mathbb{R}(D)_{10}$
$\mathbb{R}(D)_{10}$	$\mathbb{R}(D)_{11}$	$\mathbb{R}(D)_{00}$	$\mathbb{R}(D)_{01}$
$\mathbb{R}(D)_{11}$	$\mathbb{R}(D)_{10}$	$\mathbb{R}(D)_{01}$	$\mathbb{R}(D)_{00}$

Subdividing s times yields a $\mathbb{Z}_2^s$ grading. Each subdivision of the $D \times D$ matrix doubles the number of blocks in each row, and hence the non-tensorial grading is constrained by,

$$2^s \leq D$$

For comparison, the tensorial subdivision with the same number of blocks, $\underset{s}{\otimes}\mathbb{R}(2) \cong Cl(s,s;\mathbb{R})$, has double the grading, $\mathbb{Z}_2^{2s}$.

5 Conclusion

To illuminate the $\mathbb{Z}_2^n$ grading of the classical Lie algebras, the conventional classification in terms of matrix algebras over two, three or six Clifford algebras has been expanded to include a total of eight Clifford algebras. In so doing, the symplectic algebras have been placed on an equal footing with the orthogonal and unitary algebras. In particular, all the Lie algebras have been expressed in terms of pseudo-Euclidean metrics. The eight Clifford algebras and associated classical Lie algebras have gradings ranging from zero for $O(N;\mathbb{R})$ to $\mathbb{Z}_2^3$ for $Sp(2N;\mathbb{C}) \cong \mathfrak{aut}(\overline{Cl}(3,0;\mathbb{R})(N))$ and $U\!*\!(2N;\mathbb{C}) \cong \mathfrak{aut}(Cl(0,3;\mathbb{R})(N))$, as per Table I.

Lie algebras with matrix dimensions divisible by a power of two have been recast in terms of matrix algebras over higher Clifford algebras with a higher grading. For example, the $\mathbb{Z}_2^{2n+3}$ grading of $U(m\,2^n, m\,2^n;\mathbb{C})$ becomes apparent when the Lie algebra is expressed in terms of a matrix algebra over $Cl(n+1, n+2;\mathbb{R})$. The defining equations for the Lie algebras involve a matrix adjoint operator, and this generally corresponds to a non-standard adjoint of the higher Clifford algebras, ie an adjoint other than Clifford Reversion or Clifford Conjugation. It is hoped that the description of Lie algebras in terms of matrix algebras over higher Clifford algebras will facilitate the description of the associated Lie color algebras.

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[1] H. Weyl, *The Classical Groups: Their Invariants and Representations* (Princeton U.P., 1946).
[2] F. A. Berezin, *Introduction to superanalysis* (Dordrecht, Boston, 1987).
[3] R. Haag, J. T. Łopuszański, and M. Sohnius, Nucl. Phys. B **88**, 257 (1975).
[4] V. G. Kac, Adv. Math **26**, 8 (1977).
[5] V. Rittenberg and D. Wyler, Nucl. Phys. B **139**, 189 (1978).
[6] V. Rittenberg and D. Wyler, J. Math. Phys. **19**, 2193 (1978).
[7] J. Lukierski and V. Rittenberg, Phys. Rev. D **18**, 385 (1978).
[8] M. Scheunert, J. Math. Phys. **20**, 712 (1978).
[9] I. R. Porteous, *Clifford Algebras and the Classical Groups*, Cambridge Studies in Advanced Mathematics (Cambridge U.P., 1995).
[10] R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications* (Wiley, New York, 1974).
[11] E. Cartan, in Encyclopédie des sciences mathématiques **1**, 329 (1908).
[12] R. Brauer and H. Weyl, Amer. J. Math. **57**, 425 (1935).